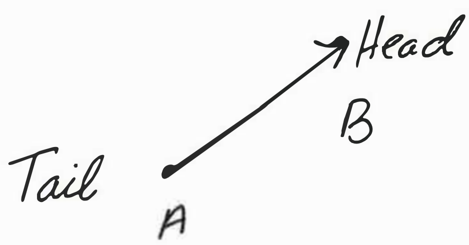
1.1. The Geometry and Algebra of Vectors

Vector: Directed line segment

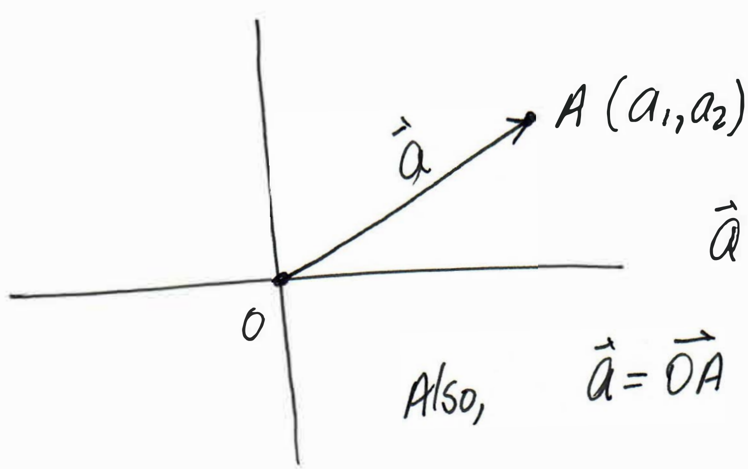
Vector from A to B in the plane



Notation: $\vec{AB}$ or $\vec{v}$

One-to-One correspondence in the plane

Points in the plane $\longrightarrow$ Vectors in the plane with tail at origin O.



$\vec{a} = [a_1, a_2]$

Also, $\vec{a} = \vec{OA}$

Position Vectors : All vectors in the plane with their tails at the origin.

Points are described by parentheses : $A(a_1, a_2)$

Vectors are described by Square brackets

$\vec{a} = [a_1, a_2]$

Components

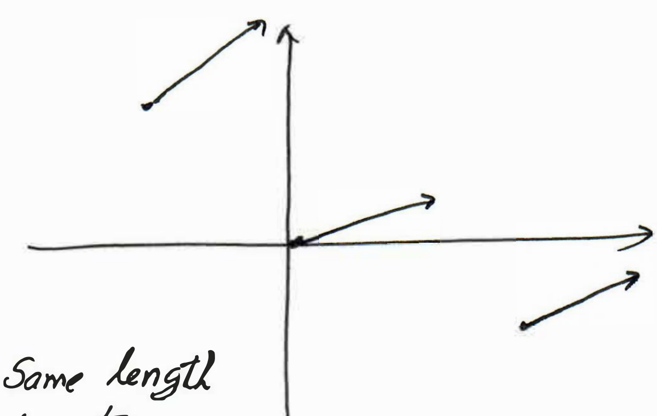
Row vectors $\rightarrow$ $\vec{a} = [a_1, a_2]$

and

Column Vectors $\longrightarrow$ $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

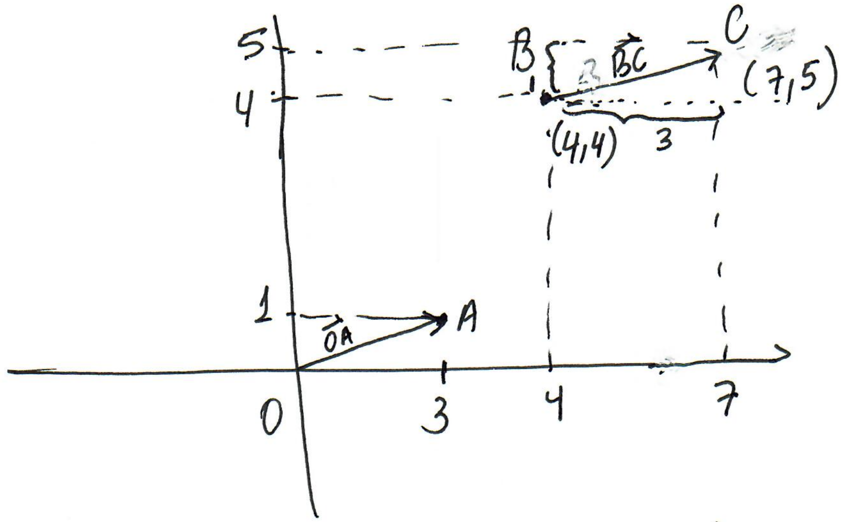
$\mathbb{R}^2 = \{$ Set of all vectors $\vec{a}$ with two components $[a_1, a_2]\}$.

Equal Vectors



They have same length and same direction.

Example:



$\vec{OA}$ is a vector in standard position

$\vec{BC}$ is $\vec{OA}$ translated to the point B (4,4).

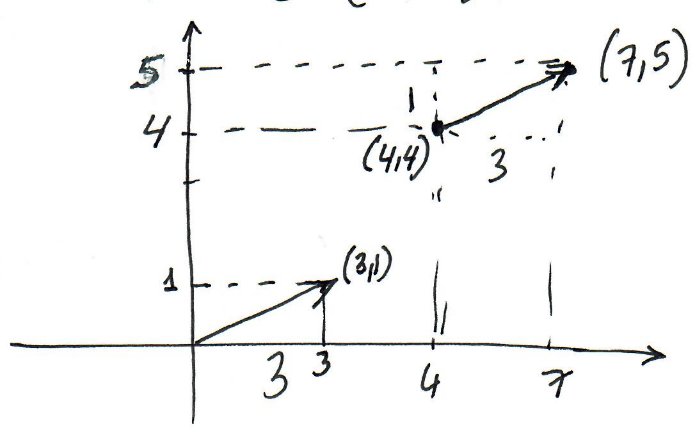
They are equal

To translate $\vec{OA}$ to the point B.

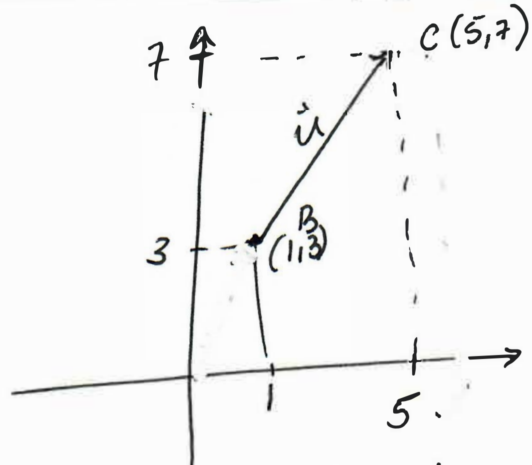
In the x-direction add 3 to 4 = 7

In the y-direction add 1 to 4 = 5

Then, we obtain: C (7,5)

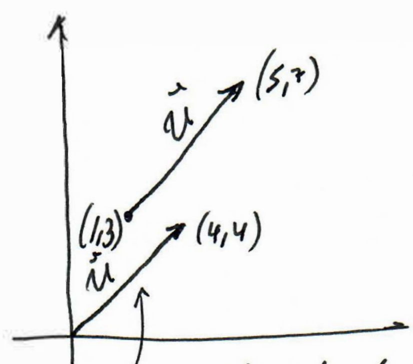


To translate a Vector $\vec{u}$ between points $B(1,3)$ and $C(5,7)$ to the origin.



You only need to Subtract $(5,7) - (1,3)$
$= (4,4)$

Then,

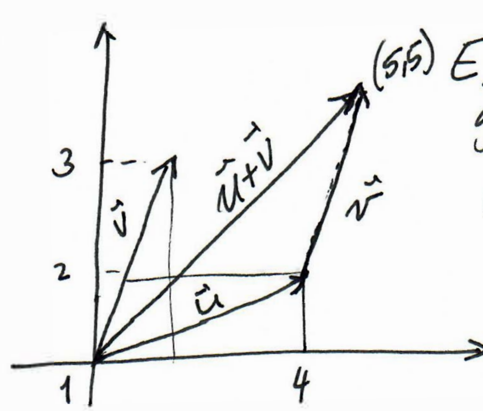


In standard position.

Vector Addition

$\vec{u} = [4, 2]$, $\quad \vec{v} = [1, 3]$

$\vec{u} + \vec{v} = [4+1, 2+3]$
$= [5, 5]$



Explain geometrical procedure

In general, Analytical def.

$$\vec{u} = [u_1, u_2], \quad \vec{v} = [\vec{v}_1, \vec{v}_2]$$

$$\vec{u} + \vec{v} \overset{def}{=\!=} [u_1 + v_1 ; u_2 + v_2]$$

<u>Geometrical Construction</u>

Head To Tail Rule

To find $\vec{u} + \vec{v}$ in the plane

① Draw $\vec{v}$ from head of $\vec{u}$

② Join by a segment tail of $\vec{u}$ with head of $\vec{v}$

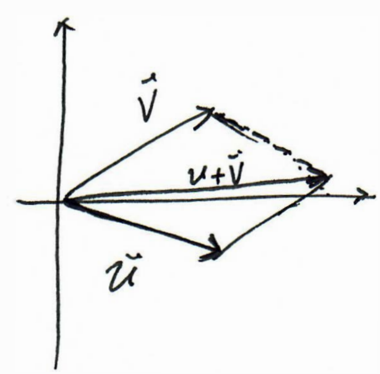
The resulting vector is $\underline{\vec{u} + \vec{v}}$

<u>Parallelogram rule for $\vec{u} + \vec{v}$</u>

① Construct the parallelogram correspondig to $\vec{u}$ and $\vec{v}$

② Draw a segment along the diagonal

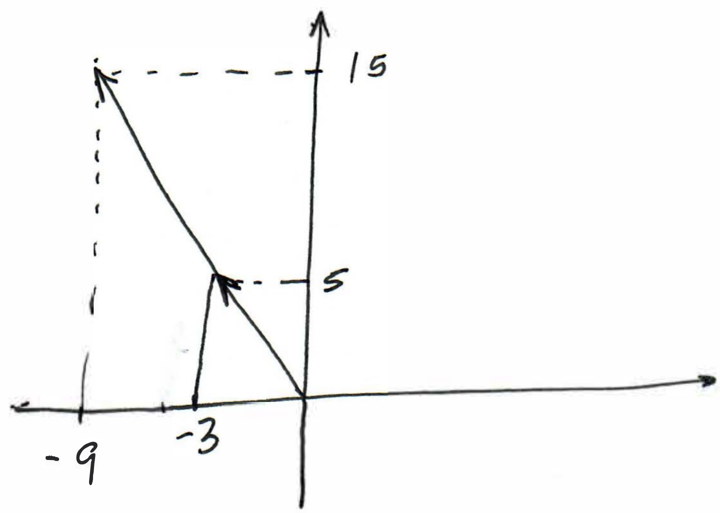
The resulting vector is $\vec{u} + \vec{v}$



Scalar Multiplication

Given $\vec{u} = [-3, 5]$ and $c = 3$

Scalar Multiplication $c\vec{u} = 3[-3,5] = [-9, 15]$



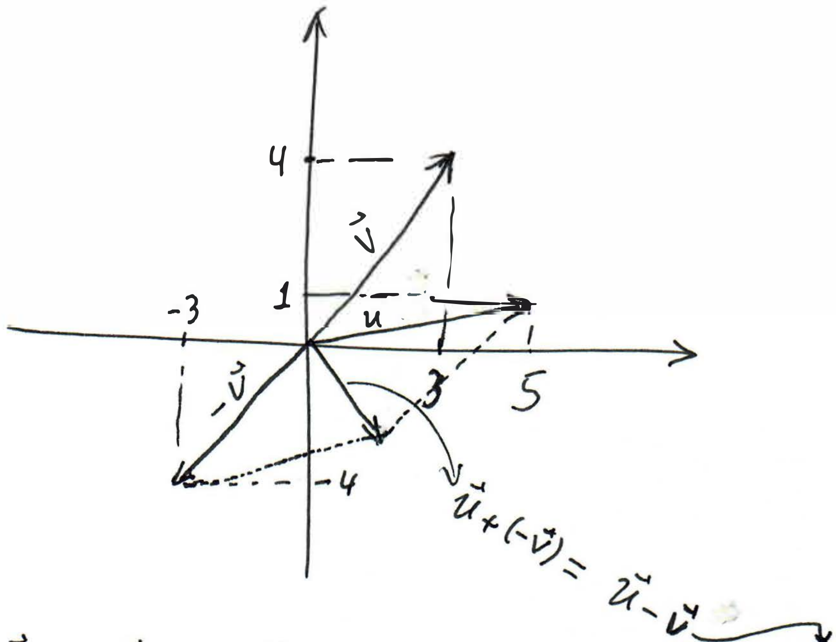
In general, if $\vec{u} = [u_1, u_2]$ and c is an arbitrary real number

$$c\vec{u} = c[u_1, u_2] = [cu_1, cu_2].$$

Vector Subtraction

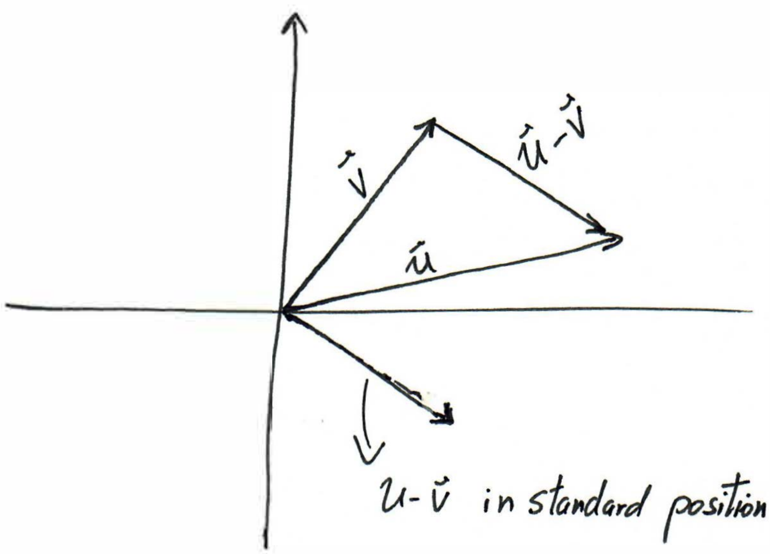
$\vec{u} = [5, 1]$, $\vec{v} = [3, 4]$

$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = [5, 1] + [-3, -4] = [2, -3]$



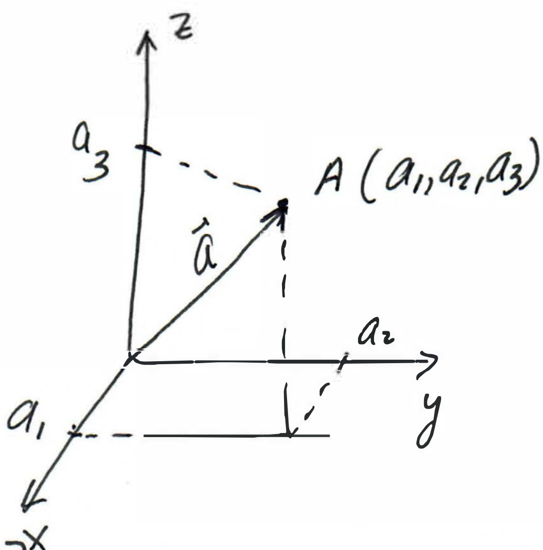
$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})=[5,1]+[-3,-4]=[2,-3]$$

Alternative (Geometrical)



Vectors in $\mathbb{R}^3$

Ordered triples of real numbers $\longrightarrow$ Vectors in the space with tail at origin.



Notation:

$\vec{a} = [a_1, a_2, a_3]$.

So

$$A(a_1, a_2, a_3) \longrightarrow \vec{a} = [a_1, a_2, a_3]$$

point $\longrightarrow$ Vector.

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Theorem 1.1 **Algebraic Properties of Vectors in $\mathbb{R}^n$**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutativity
b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associativity
c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributivity
f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributivity
g. $c(d\mathbf{u}) = (cd)\mathbf{u}$
h. $1\mathbf{u} = \mathbf{u}$

The word *theorem* is derived from the Greek word *theorema*, which in turn comes from a word meaning "to look at." Thus, a theorem is based on the insights we have when we look at examples and extract from them properties that we try to prove hold in general. Similarly, when we understand something in mathematics—the proof of a theorem, for example—we often say, "I see."

Remarks

- Properties (c) and (d) together with the commutativity property (a) imply that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $-\mathbf{u} + \mathbf{u} = \mathbf{0}$ as well.
- If we read the distributivity properties (e) and (f) from right to left, they say that we can *factor* a common scalar or a common vector from a sum.

Proof We prove properties (a) and (b) and leave the proofs of the remaining properties as exercises. Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$, $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$.

(a)
$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]\\ &= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]\\ &= [v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n]\\ &= [v_1, v_2, \ldots, v_n] + [u_1, u_2, \ldots, u_n]\\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

The second and fourth equalities are by the definition of vector addition, and the third equality is by the commutativity of addition of real numbers.

(b) Figure 1.18 illustrates associativity in $\mathbb{R}^2$. Algebraically, we have

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n]\\ &= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n]\\ &= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n]\\ &= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)]\\ &= [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]\\ &= [u_1, u_2, \ldots, u_n] + ([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n])\\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

The fourth equality is by the associativity of addition of real numbers. Note the careful use of parentheses.

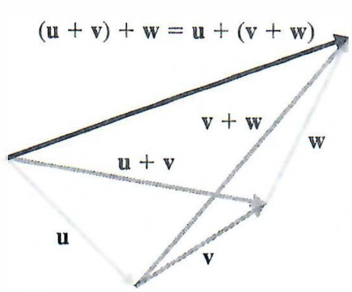

Figure 1.18